MAT 3361, INTRODUCTION TO MATHEMATICAL LOGIC, Fall 2004

Lecture Notes: Analytic Tableaux

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1 Analytic Tableaux

Definition. A *signed formula* is an expression TX or FX, where X is an (unsigned) formula. Under a given valuation, a signed formula TX is called *true* if X is true, and *false* if X is false. Also, a signed formula FX is called *true* if X is false, and *false* if X is true.

We begin with the following observations about signed formulas:

Observation 1.1. *For all propositions X, Y:*

1a. $T(\neg X) \Rightarrow FX$.
1b. $F(\neg X) \Rightarrow TX$.
2a. $T(X \wedge Y) \Rightarrow TX$ *and* TY.
2b. $F(X \wedge Y) \Rightarrow FX$ *or* FY.
3a. $T(X \vee Y) \Rightarrow TX$ *or* TY.
3b. $F(X \vee Y) \Rightarrow FX$ *and* FY.
4a. $T(X \to Y) \Rightarrow FX$ *or* TY.
4b. $F(X \to Y) \Rightarrow TX$ *and* FY.

The method of analytic tableaux can be summarized as follows: To prove the validity of a proposition X, we assume FX and derive a contradiction, using the rules from Observation 1.1. In doing so, we follow a specific format which is illustrated in the following example.

Example 1.2. An analytic tableau proving the validity of $X = (p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$ is shown in Table 1. Note: the line numbers, such as (1), (2) etc, are not part of the formalism; they are only used for our discussion.

The initial premise on line (1) is of the form $F(Y \to Z)$, where $Y = (p \vee (q \wedge r))$ and $Z = ((p \vee q) \wedge (p \vee r))$. By rule 4b, we can conclude both TY and FZ.

(1) $F(p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$
(2) $T(p \vee (q \wedge r))$
(3) $F((p \vee q) \wedge (p \vee r))$

(4) Tp		(11) $T(q \wedge r)$	
		(12) Tq	
		(13) Tr	
(5) $F(p \vee q)$	(8) $F(p \vee r)$	(14) $F(p \vee q)$	(17) $F(p \vee r)$
(6) Fp	(9) Fp	(15) Fp	(18) Fp
(7) Fq	(10) Fr	(16) Fq	(19) Fr
$\times$	$\times$	$\times$	$\times$

Table 1: An analytic tableau for $X = (p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$.

These are called the *direct consequences* of line (1), and we write them in lines (2) and (3), respectively. After we have done so, we say that line (1) has been *used*.

Now consider the formula in line (2), which is of the form $T(X \vee Y)$, where $X = p$ and $Y = q \wedge r$. From rule 3a, we may conclude that *either* TX *or* TY holds. Since the conclusion in this cases involves a choice between two possibilities, we say that the formula on line (2) *branches*. When using such a formula, the tableaux splits into two branches, one for each possibility. This has been done in lines (4) and (11).

Continuing in a similar fashion, we use up all the lines containing composite formulas. We say that a branch is *closed* if it contains both TX and FX, for some signed formula X. We mark closed branches with the symbol $\times$. A closed branch represents a contradiction. Since in this example, all branches are closed, we conclude that the original formula $(p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$ is valid.

As the example shows, note that there are essentially two types of signed formulas:

(A) signed formulas with direct consequences, which are $T(\neg X)$, $F(\neg X)$, $T(X \wedge Y)$, $F(X \vee Y)$, and $F(X \to Y)$, and

(B) signed formulas which branch, which are $F(X \wedge Y)$, $T(X \vee Y)$, and $T(X \to Y)$.

When using a formula of type (A), we simply add all of its direct consequences to each branch underneath the formula being used. When using a formula of type (B), we split each branch underneath the formula into two new branches. The rules for tableaux can be summarized schematically as follows:

$$\frac{T(\neg X)}{FX} \qquad \frac{F(\neg X)}{TX}$$

$$\frac{T(X \wedge Y)}{\begin{array}{c}TX\\TY\end{array}} \qquad \frac{F(X \wedge Y)}{FX \mid FY}$$

$$\frac{T(X \vee Y)}{TX \mid TY} \qquad \frac{F(X \vee Y)}{\begin{array}{c}FX\\FY\end{array}}$$

$$\frac{T(X \rightarrow Y)}{FX \mid TY} \qquad \frac{F(X \rightarrow Y)}{\begin{array}{c}TX\\FY\end{array}}$$

Definition. A branch is said to be *complete* if every formula on it has been used. A tableau is said to be *completed* if every one of its branches is complete or closed. A tableau is said to be *closed* if all of its branches are closed. A tableau is said to be *open* if it is not closed, i.e., if it has at least one open branch.

We say that a formula X has been *proved by the tableaux method* if there exists a closed analytic tableau with origin FX.

Strategies. Our goal is to find a completed analytic tableau for a given formula. There are different strategies for deriving such a tableau.

Strategy 1 is to work systematically downwards: in this strategy, we never use a line until all lines above it have been used. When using this strategy, we are guaranteed to arrive at a completed tableau after a finite number of steps. However, strategy 1 is often more inefficient than the following strategy 2:

Strategy 2: give priority to lines of type (A). This means that we use up all lines of type (A) before using those of type (B). When following this strategy, we postpone the creation of new branches until absolutely necessary, thus keeping the size of the tableau smaller when compared to strategy 1.

Abbreviations. We often use the following shortcut notation when discussion signed formulas: We use the letter α to stand for any signed formula of type (A). In this case, we use α_1 and α_2 to denote the direct consequences (in the special case where there is only one direct consequence, we will set $\alpha_1 = \alpha_2$). All possibilities for α, α_1, and α_2 are summarized in the following table:

α	α_1	α_2
$T(X \wedge Y)$	TX	TY
$F(X \vee Y)$	FX	FY
$F(X \to Y)$	TX	FY
$T(\neg X)$	FX	FX
$F(\neg X)$	TX	TX

We also use the letter β to stand for any signed formula of type (B). In this case, we use β_1 and β_2 to denote the two alternative consequences. For reasons of symmetry, we further also allow β to also stand for a signed formula which is a negation, in which case we set $\beta_1 = \beta_2$. Thus, all possibilities for β, β_1, β_2 are summarized as follows:

β	β_1	β_2
$F(X \wedge Y)$	FX	FY
$T(X \vee Y)$	TX	TY
$T(X \to Y)$	FX	TY
$T(\neg X)$	FX	FX
$F(\neg X)$	TX	TX

With these conventions, the rules for tableaux can be written succinctly as follows:

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad\qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

Definition. The *conjugate* of a signed formula FX is TX, and the conjugate of a signed formula TX is FX. We write $\bar{\varphi}$ for the conjugate of a signed formula φ.

We also observe the following: the conjugate of any α is some β, and in this case, $\overline{(\alpha_1)} = \beta_1$ and $\overline{(\alpha_2)} = \beta_2$. The conjugate of any β is some α, and in this case, $\overline{(\beta_1)} = \alpha_1$ and $\overline{(\beta_2)} = \alpha_2$. Moreover, for any signed formula φ, we have $\bar{\bar{\varphi}} = \varphi$.

2 Soundness and Completeness for Analytic Tableaux

Recall that we have called a branch of a tableau "complete" if every formula on it "has been used". With our convention on using the letters α and β for signed formulas, we may express this more precisely:

A branch θ of a tableau $\mathcal{T}$ is *complete* if for every $\alpha \in \theta$, both $\alpha_1, \alpha_2 \in \theta$, and for every $\beta \in \theta$, either $\beta_1 \in \theta$ or $\beta_2 \in \theta$.

As before, we say that a tableau $\mathcal{T}$ is *completed* if every branch θ of $\mathcal{T}$ is either closed or complete.

2.1 Tableaux and valuations

Let $[\![-]\!]$ be a valuation. We extend $[\![-]\!]$ to signed formulas in the obvious way by letting $[\![TX]\!] = [\![X]\!]$ and $[\![FX]\!] = 1 - [\![X]\!]$. Thus, FX is true under a given valuation iff X is false under that valuation.

Definition. Let $[\![-]\!]$ be a valuation. We say that a branch θ of a tableau $\mathcal{T}$ is *true* under $[\![-]\!]$ if for all $\varphi \in \theta$, $[\![\varphi]\!] = 1$. We say that $\mathcal{T}$ is *true* under $[\![-]\!]$ if there is at least one branch θ of $\mathcal{T}$ such that θ is true under $[\![-]\!]$.

2.2 Soundness

Soundness states that if a formula X is provable by the tableaux method, then X is a tautology.

Theorem 2.1 (Soundness). *Suppose X is a proposition, and $\mathcal{T}$ is a closed tableau with origin FX. Then X is a tautology.*

The proof depends on the following lemma:

Lemma 2.2. *Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are tableaux such that $\mathcal{T}_2$ is an immediate extension of $\mathcal{T}_1$. Then $\mathcal{T}_2$ is true under every interpretation under which $\mathcal{T}_1$ is true.*

Proof. Suppose $\mathcal{T}_1$ is true under the given valuation $[\![-]\!]$. Then $\mathcal{T}_1$ has at least one true branch θ. Now $\mathcal{T}_2$ was obtained by adding one or two successors to the endpoint of some branch θ_1 of $\mathcal{T}_1$. If $\theta_1 \neq \theta$, then θ is still a branch of $\mathcal{T}_2$, hence $\mathcal{T}_2$ is true and we are done. Assume therefore that $\theta_1 = \theta$. Then θ was extended by one of the following operations:

(A) For some $\alpha \in \theta$, we have added α_1 or α_2, so $\theta \cup \{\alpha_1\}$ or $\theta \cup \{\alpha_2\}$ is a branch of $\mathcal{T}_2$. But $[\![\alpha]\!] = 1$, therefore $[\![\alpha_1]\!] = 1$ and $[\![\alpha_2]\!] = 1$, therefore $\mathcal{T}_2$ contains a true branch.

(B) For some $\beta \in \theta$, we have added both β_1 and β_2, so both $\theta \cup \{\beta_1\}$ and $\theta \cup \{\beta_2\}$ are branches of $\mathcal{T}_2$. But $[\![\beta]\!] = 1$, therefore $[\![\beta_1]\!] = 1$ or $[\![\beta_2]\!] = 1$, therefore $\mathcal{T}_2$ contains at least one true branch. □

Lemma 2.3. *Let $[\![-]\!]$ be a fixed valuation. For any tableau $\mathcal{T}$, if the origin of $\mathcal{T}$ is true under $[\![-]\!]$, then $\mathcal{T}$ is true under $[\![-]\!]$.*

Proof. This is an immediate consequence of the previous lemma, by induction: $\mathcal{T}$ is obtained from the origin by repeatedly extending the tableau in the sense of Lemma 2.2, at each step preserving truth. □

Proof of the Soundness Theorem: Let $\mathcal{T}$ be a closed tableau with origin FX, and let $[\![-]\!]$ be any valuation. Since $\mathcal{T}$ is closed, each branch contains some formula and its negation, and therefore $\mathcal{T}$ cannot be true under $[\![-]\!]$. From Lemma 2.3, it follows that the origin of $\mathcal{T}$ is false under $[\![-]\!]$, thus $[\![FX]\!] = 0$, thus $[\![X]\!] = 1$. Since $[\![-]\!]$ was arbitrary, it follows that X is a tautology. □

2.3 Completeness

Completeness is the converse of soundness: it states that if X is a tautology, then X is provable by the tableaux method. In fact we will prove something slightly stronger, namely, if X is a tautology, then *every* strategy for completing a tableaux for X will lead to a closed tableaux.

Theorem 2.4 (Completeness). *(a) Suppose X is a tautology. Then every completed tableau with origin FX must be closed.*

(b) Suppose X is a tautology. Then X is provable by the tableaux method.

The main ingredient in the proof is the notion of a Hintikka set.

Definition. Let S be a (finite or infinite) set of signed formulas. Then S is called a *Hintikka set* (or *downward saturated*) if it satisfies the following three conditions:

(H_0) There is no propositional variable p such that both $Tp \in S$ and $Fp \in S$.

(H_1) If $\alpha \in S$, then $\alpha_1 \in S$ and $\alpha_2 \in S$.

(H_2) If $\beta \in S$, then $\beta_1 \in S$ or $\beta_2 \in S$.

Note that, by definition, a complete non-closed branch θ is a Hintikka set.

If S is a set of signed formulas, we say that S is *satisfiable* if there exists a valuation $[\![-]\!]$ such that for all $\varphi \in S$, $[\![\varphi]\!] = 1$.

Lemma 2.5 (Hintikka Lemma). *Every Hintikka set is satisfiable.*

Proof. Let S be a Hintikka set, and define a valuation as follows: for any propositional variable p, let

$$\begin{array}{ll} [\![p]\!] = 1 & \text{if } Tp \in S, \\ [\![p]\!] = 0 & \text{if } Fp \in S, \\ [\![p]\!] = 1 & \text{if } Tp \notin S \text{ and } Fp \notin S. \end{array}$$

Note that, since S is a Hintikka set, we cannot have $Tp \in S$ and $Fp \in S$ at the same time. Thus, this is well-defined. We recursively extend $[\![-]\!]$ to composite formulas in the unique way.

We now claim that for all $\varphi \in S$, $[\![\varphi]\!] = 1$. This is proved by induction on φ. For atomic φ, this is true by definition. If φ is composite, then there are two cases:

(A) φ is some α. Then by (H_1), $\alpha_1 \in S$ and $\alpha_2 \in S$. By induction hypothesis, $[\![\alpha_1]\!] = 1$ and $[\![\alpha_2]\!] = 1$, therefore $[\![\alpha]\!] = 1$.

(B) φ is some β. Then by (H_2), $\beta_1 \in S$ or $\beta_2 \in S$. By induction hypothesis, $[\![\beta_1]\!] = 1$ or $[\![\beta_2]\!] = 1$, therefore $[\![\beta]\!] = 1$.

Thus, $[\![\varphi]\!] = 1$ for all $\varphi \in S$, and hence S is satisfiable as desired. □

Proof of the Completeness Theorem:

(a) Suppose X is a tautology, and $\mathcal{T}$ is some completed tableau with origin FX. Suppose θ is some branch of $\mathcal{T}$ which is not closed. Then θ is a Hintikka set by definition, hence satisfiable by the Hintikka Lemma. Thus, there exists some valuation $[\![-]\!]$ which makes θ true. Since $FX \in \theta$, we have $[\![FX]\!] = 1$, hence $[\![X]\!] = 0$, hence X is not a tautology, a contradiction. It follows that every branch of $\mathcal{T}$ is closed.

(b) It is easy to see that for any signed formula φ, there exists a completed tableau with origin φ. For example, such a tableau is obtained by following

Strategy 1 or Strategy 2 from Section 1. In particular, if X is a tautology, then there exists a completed tableau with origin FX, which is closed by (a), and hence X is provable by the tableaux method. □

2.4 Discussion of the proofs

We note the following features of the soundness and completeness proofs:

Soundness proof. The proof of soundness essentially proceeds by induction on tableaux, as is evident in the proof of Lemma 2.3. One fixes a valuation, then proves by induction that all derivations respect the given valuation.

This proof method is typical of soundness proofs in general. Compare this proof e.g. to the soundness proof for natural deduction in Lemma 1.5.1 of van Dalen's book. Most of the time, soundness proofs are relatively easy.

Completeness proof. The central part of any completeness proof is a satisfiability result: for a certain set of formulas, one must show that there exists a valuation making all the formulas true. To see why this is central, notice that the completeness property can be equivalently expressed as follows:

If X is *not* provable, then X is *not* a tautology.

Thus, it is natural to start by assuming that X is not provable (e.g., its analytic tableau does not close). Now one must prove that X is not a tautology, which amounts to finding a specific valuation which makes X false. In the case of analytic tableaux, this valuation is obtained using Hintikka's lemma.

Compare this to the completeness proof for natural deduction in Section 1.5 of van Dalen's book. It uses a completely different method, yet the central lemma is the one which allows one to construct a valuation, namely Lemma 1.5.11 (every consistent set is satisfiable). The method used for constructing a suitable valuation varies from proof system to proof system, and usually gets more difficult as features are added to the logic.